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EFFECT OF ELASTICITY ON THE DYNAMICS OF A SUPERCONDUCTING ROTOR ROTATING IN A MAGNETIC FIELD

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A spherical shape of the outer surface of rotors of some types of noncontact gyroscopes gives rise to conditions, where the force field ensures the stability of the center of mass relative to the base and has an insignificant effect on the angular motion of the rotor. However, there are some effects (for instance, the Barnett–London effect), which lead to emergence of moments of mechanical forces even for spherical bodies. The effect of rotor elasticity on the motion of a superconducting deformable spherical solid body in a magnetic field is studies. It is shown that the moment of mechanical forces acting on the body in the magnetic field is proportional in the first approximation to the angular velocity squared. The effect of this moment on the dynamics of angular motion of the rotor is studied.

1. Effect of the Magnetic Field on a Rotating Superconducting Rotor. If the body is not rotating, then the magnetic field $\boldsymbol{H}$ in the ambient space is a superposition of the fields generated by the sources $\boldsymbol{H}_0$ and image currents $\boldsymbol{H}_{\rm im}$ in a thin surface layer of the superconductor and satisfies the following equations and conditions at the outer surface of the rotor S_1 and far from it [1]:

$$\operatorname{rot}\boldsymbol{H} = 0, \quad \operatorname{div}\boldsymbol{H} = 0, \quad \boldsymbol{H}\cdot\boldsymbol{n}_0^{(1)} = 0 \ \text{ on } \ S_1, \quad \boldsymbol{H}_{\rm im} \to 0 \ \text{ as } \ |\boldsymbol{r}| \to \infty. \tag{1.1}$$

Here $\boldsymbol{n}_0^{(1)}$ is the vector normal to the surface S_1.

Noncontact retention of the body is ensured by the magnetic pressure $\boldsymbol{P} = (H^2/(8\pi))\boldsymbol{n}_0^{(1)}$, which is caused by the discontinuity of the components of the Maxwell stress tensor on S_1. The magnetic pressure is the reason for rotor deformation. Depending on the magnitude of the magnetic field and elastic properties of the body, this deformation is either ignored as being small or is taken into account as a certain small initial deviation of the outer surface of the rotor from the spherical shape.

During rotation, the rotor is deformed, which leads to the disturbance of the magnetic field $\boldsymbol{H}_{\rm im}$ characterized by the vector $\boldsymbol{H}_1$, for which we have the problem

$$\begin{gathered}\operatorname{rot}\boldsymbol{H}_1 = 0, \quad \operatorname{div}\boldsymbol{H}_1 = 0, \quad \boldsymbol{H}_1\cdot\boldsymbol{n}_0^{(1)} + \boldsymbol{H}\cdot\boldsymbol{n}_1^{(1)} = 0 \quad \text{on} \quad S_1,\\ \boldsymbol{H}_1 \to 0 \quad \text{as} \quad |\boldsymbol{r}| \to \infty,\end{gathered} \tag{1.2}$$

where $\boldsymbol{n}_1^{(1)}$ is a correction to $\boldsymbol{n}_0^{(1)}$ that is caused by the disturbance of the body surface due to rotation.

In addition to the coordinate system X_i ($i = 1, 2, 3$) attached to the sources of the magnetic field, in which we calculate $\boldsymbol{H}$ and $\boldsymbol{H}_1$, we introduce the coordinate system Z_i. The common origin of these coordinate systems is a fixed point (center of mass). The body as a whole does not move in the coordinate system Z_i rotating with an angular velocity $\boldsymbol{\omega}$. Therefore, the following integral relations are valid:

$$\int\limits_V \boldsymbol{U}\,dV = 0, \qquad \int\limits_V \boldsymbol{r}\times\boldsymbol{U}\,dV = 0 \tag{1.3}$$

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($\boldsymbol{U}$ is the strain vector). Hereinafter we use dimensionless variables and parameters. The scale factors for the corresponding physical quantities are the rotor radius R, the characteristic time of its motion as a whole t_* (rotation of the coordinate system Z_i relative to X_i), and the parameter H_* characterizing the external magnetic field.

The rotor has a cavity designed to decrease the mass of the body suspended in the magnetic field and ensure its rotation in a prescribed direction (the relation between the moments of inertia in the initial state is determined by the equality $I_{11}^0 = I_{22}^0 = I_{33}^0$).

The equations of motion of the volume element and the conditions on the rotor surface in the coordinate system Z_i have the following form:

$$(\chi + 1)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{U} + \Delta\boldsymbol{U} = \varepsilon(\ddot{\boldsymbol{U}} + \boldsymbol{F}), \tag{1.4}$$

$$(\sigma_{ij} - \delta T_{ij})(n_{0j}^{(1)} + n_{1j}^{(1)}) = 0 \quad \text{on} \quad S_1, \qquad \sigma_{ij}(n_{0j}^{(2)} + n_{1j}^{(2)}) = 0 \quad \text{on} \quad S_2. \tag{1.5}$$

Here $\chi = \lambda/\mu$ (λ and μ are the Lamé constants), $\varepsilon = \rho R^2/(\mu t_*^2)$, ρ is the density, $\delta = H_*^2/\mu$, $\boldsymbol{F} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + 2\dot{\boldsymbol{\omega}} \times \dot{\boldsymbol{U}}$ is the vector of inertia forces, σ_{ij} is the stress tensor, $\boldsymbol{n}^{(2)} = \boldsymbol{n}_0^{(2)} + \boldsymbol{n}_1^{(2)}$ is the vector normal to the inner surface S_2, and $T_{ij} = (1/(4\pi))(H_iH_j - \delta_{ij}H^2/2)$.

Generally speaking, the rotor experiences also the action of nonelectromagnetic surface and volume forces, which should be reflected in Eqs. (1.4) and (1.5). The force of gravity and the resistance of the ambient medium may serve as examples. We assume that the effect of these forces on rotor deformation is negligibly small.

We consider only the angular motion of the rotor, which is possible if the frequencies of angular motion are significantly smaller than the lower frequency of elastic eigenoscillations of the body (i.e., the inequality $\varepsilon \ll 1$ is valid) [2].

We represent the strain vector $\boldsymbol{U}(\boldsymbol{r}, t)$ as a series in the parameters ε and δ:

$$\boldsymbol{U} = \varepsilon\boldsymbol{U}_1 + \delta\varepsilon\boldsymbol{U}_2 + \dots\ . \tag{1.6}$$

Substituting $\boldsymbol{U}$ into (1.3)–(1.5), we obtain the problem

$$\begin{gathered}\int\limits_V \boldsymbol{U}_1\,dV = 0, \quad \int\limits_V \boldsymbol{r} \times \boldsymbol{H}_1\,dV = 0, \quad (\chi + 1)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{U}_1 + \Delta\boldsymbol{U}_1 = \boldsymbol{F}^{(1)},\\ \sigma_{ij}^{(1)} n_{0j}^{(k)} = 0 \quad \text{on} \quad S_k \qquad (k = 1,\ 2),\end{gathered} \tag{1.7}$$

where $\sigma_{ij}^{(1)} = \sigma_{ij}(U_{1m})$ is the stress tensor calculated on the basis of $\boldsymbol{U}_1$ and $\boldsymbol{F}^{(1)} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}$.

In accordance with (1.7), the first term of series (1.6) is the deformation of the freely rotating body in the absence of the magnetic field.

To find the vector $\boldsymbol{U}_2(r, t)$, one has to consider together the electrodynamic equations for $\boldsymbol{H}$ and $\boldsymbol{H}_1 = \varepsilon\boldsymbol{h}$ and the system of equations

$$\int\limits_V \boldsymbol{U}_2\,dV = 0, \quad \int\limits_V \boldsymbol{r} \times \boldsymbol{H}_2\,dV = 0, \quad (\chi + 1)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{U}_2 + \Delta\boldsymbol{U}_2 = 0,$$

$$(\sigma_{ij}^{(2)} - t_{ij})n_{0j}^{(1)} - T_{ij}^{(0)} n_{1j}^{(1)} = 0 \quad \text{on} \quad S_1, \qquad \sigma_{ij}^{(2)} n_{0j}^{(2)} = 0 \quad \text{on} \quad S_2,$$

where $\sigma_{ij}^{(2)} = \sigma_{ij}(U_{2m})$, $n_{1j}^{(1)} = \varepsilon\Delta n_j(U_{1m})$, and $t_{ij} = (1/(4\pi))(H_ih_j + H_jh_i - \delta_{ij}\boldsymbol{h} \cdot \boldsymbol{H})$.

It is assumed that the outer surface of the body in the nondeformed state is spherical. If the surface is not spherical, system (1.7) remains unchanged, and S_1 should be considered as a spherical surface in calculating $\boldsymbol{U}_1$. The effect of nonsphericity is manifested in calculating the subsequent terms of series (1.6).

The motion of the rotor is determined by the vector $\boldsymbol{H}(r, t)$ and the angular velocity $\boldsymbol{\omega}(t)$. System (1.7) has the solutions $\boldsymbol{U}_1$ and $\boldsymbol{\omega}$ corresponding to free motion of a perfectly rigid body with the main moment of inertia I_{ii}^0. If we consider the following approximation in terms of the small parameters of the problem, we can find $\boldsymbol{U}_2$ and the correction to $\boldsymbol{\omega}$, which takes into account the elastic properties of the rotor. However, it is also possible to derive the equations for the components of the angular velocity without solving this complicated problem.

2. Equations of Motion of the Deformable Rotor. We perform vector multiplication of (1.4) by $\boldsymbol{r} + \boldsymbol{U}$ and integrate it over the body volume, taking into account conditions (1.5). Substituting εU_1 instead of U into the resultant equation, we obtain

$$\dot{\boldsymbol K} + \boldsymbol\omega \times \boldsymbol K = \boldsymbol M(\boldsymbol H), \tag{2.1}$$

where $K_i = I_{ii}^0\omega_i + \varepsilon I_{ij}^{(1)}\omega_j$ are the components of the kinetic moment vector, $I_{ij}^1 = 2\int\limits_V (r_k U_{1k}\delta_{ij} - r_i U_{1j})\,dV$, and $\boldsymbol M$ is the moment of forces caused by deformation and the magnetic field.

We transform Eq. (2.1) to the form, which it would have in the case of a perfectly rigid body:

$$\boldsymbol K^{(0)} + \boldsymbol\omega \times \boldsymbol K^{(0)} = \boldsymbol M_{\rm el} + \boldsymbol M(\boldsymbol H), \qquad K_i^{(0)} = I_{ij}^{(0)}\omega_j. \tag{2.2}$$

The transition from a deformable rotor to the model of a perfectly rigid body is natural from the viewpoint of engineering calculations of gyroscope motion and processing of experimental data. Rotor elasticity is taken into account by the moment of forces $\boldsymbol M_{\rm el}$ in the right part of Eq. (2.2), which depends only on elastic properties of the body and angular velocity. The moment $\boldsymbol M(\boldsymbol H)$ is caused by deformation of the rotating rotor in an undisturbed field and is written as

$$\boldsymbol M = \delta \int\limits_{S_1} \boldsymbol r \times \boldsymbol T\,dS.$$

Thus, the problem of angular motion of a deformable superconducting rotor in a magnetic field reduces to studying the angular motion of a perfectly rigid body under the action of the moments $\boldsymbol M_{\rm el}$ and $\boldsymbol M(\boldsymbol H)$. To calculate them, one has to know only the deformation of the body during its rotation in the absence of the magnetic field.

3. Angular Motion of the Deformable Rotor in a Magnetic Field. We assume that the external field is axisymmetric, and the OX_3 axis coincides with the axis of suspension. In the approximation considered, Z_i is a coordinate system rigidly attached to the perfectly rigid body; the axes are directed along the main axes of the ellipsoid of inertia ($I_{11}^0 \approx I_{22}^0 \approx I_{33}^0$). We also introduce a coordinate system Y_i attached to the kinetic moment vector of the perfectly rigid body; the directions OY_3 and $\boldsymbol K^{(0)}$ coincide.

We study the rotor motion, which is interpreted as the motion of the kinetic moment vector $\boldsymbol K^{(0)}$ relative to X_i, and the superimposed motion of the body (coordinate system Z_i) relative to $\boldsymbol K^{(0)}$. The equation of motion of the vector $\boldsymbol K^{(0)}$ in the coordinate system X_i has the form $d\boldsymbol K^0/dt = \boldsymbol M_{\rm el} + \boldsymbol M(\boldsymbol H)$.

The position of the coordinate system Z_i relative to Y_i is set by the Euler angles. Then, the rotor motion relative to the kinetic moment vector is determined by the following system:

$$\dot\alpha = K^{(0)}\Big(\frac{\cos^2\gamma}{I_{11}^0} + \frac{\sin^2\gamma}{I_{22}^0}\Big) - \frac{1}{K^{(0)}}\Big(\cot\rho\,\frac{dV}{d\rho} + \cot\beta\,\frac{dV}{d\beta}\Big),$$

$$\dot\beta = K^{(0)}\Big(\frac{1}{I_{22}^0} - \frac{1}{I_{11}^0}\Big)\sin\beta\sin\gamma\cos\gamma + \frac{1}{K^{(0)}\sin\beta}\Big(\cos\beta\,\frac{\partial V}{\partial\alpha} - \frac{\partial V}{\partial\gamma}\Big),$$

$$\dot\gamma = K^{(0)}\cos\beta\Big(\frac{1}{I_{33}^0} - \frac{\cos^2\gamma}{I_{11}^0} - \frac{\sin^2\gamma}{I_{22}^0}\Big) + \frac{1}{K^{(0)}\sin\beta}\,\frac{\partial V}{\partial\beta}.$$

Here α, β, and γ are the angles of precession, nutation, and own rotation, respectively, and ρ is the angle between the OX_3 and OY_3 axes.

We represent the force function V corresponding to the moments $\boldsymbol M_{\rm el}$ and $\boldsymbol M(\boldsymbol H)$ for the rotor rotating in an axisymmetric field as a scalar product of irreducible tensors [3, 4], which is convenient for solving the problem. It takes into account the increment of the kinetic energy due to the own elasticity of the body and the energy of interaction of the undisturbed field with the body deformed in the course of rotation, which had a spherical surface in the initial state.

Elastic properties of the body taken into account, its kinetic energy $T_0 = I_{ii}^0\omega_i^2/2$ increases by $T_{\rm el} = \varepsilon\alpha_{ijkl}\omega_i\omega_j\omega_k\omega_l$. From four vectors $\boldsymbol\omega$, we can compose only the scalar ω^4 and irreducible tensors of the second and fourth ranks $\omega^2\{\boldsymbol\omega_1 \otimes \boldsymbol\omega_1\}_2$ and $\{\{\{\boldsymbol\omega_1 \otimes \boldsymbol\omega_1\}_2 \otimes \boldsymbol\omega_1\}_3 \otimes \boldsymbol\omega_1\}_4$, which are expressed through the spherical functions [3]:

$$\{\boldsymbol\omega_1 \otimes \boldsymbol\omega_1\}_{2m} = \omega^2\sqrt{\frac{2}{3}}\,Y_{2m}(\boldsymbol e), \quad \{\{\{\boldsymbol\omega_1 \otimes \boldsymbol\omega_1\}_2 \otimes \boldsymbol\omega_1\}_3 \otimes \boldsymbol\omega_1\}_{4m} = 2\sqrt{\frac{2}{35}}\,\omega^4 Y_{4m}(\boldsymbol e).$$

Then, the energy $T_{\rm el}$ may be represented as

$$T_{\rm el} = \varepsilon\omega^4[a_0 + a_2 \cdot Y_2(\boldsymbol{e}) + a_4 \cdot Y_4(\boldsymbol{e})]. \tag{3.1}$$

Here $Y_{lm}(\boldsymbol{e})$ is a spherical function determined without the term $\sqrt{2l+1}/(4\pi)$, $\boldsymbol{e}$ is a unit vector directed along $\boldsymbol{\omega}$, a_0 is a scalar, a_2 and a_4 are, respectively, the irreducible tensors of the second and fourth ranks, which consist of the components of the tensor α_{ijkl}, and $\{P_n \otimes q_n\}_s$ is the tensor product of two irreducible tensors of the same rank [3].

The generic expression for $T_{\rm el}$ includes 15 independent components, since the tensors a_2 and a_4 have five and nine components, respectively. The symmetry of the body may reduce the number of independent components of $T_{\rm el}$ [5].

The relationship between the moments of inertia I^0_{ii} is determined by the internal cavity in the rotor. The quantities a_0, a_2, and a_4 are functions of the parameters that characterize the cavity and, hence, depend on the moments of inertia. Since the main moments are close in magnitude in our case, we may confine ourselves to the first term in Eq. (3.1), because the order of the parameters a_2 and a_4 is determined (in contrast to a_0) by the difference in the moments of inertia.

The energy of interaction of the undisturbed field and the elastic body, which depends on the moment $\boldsymbol{M}_2$, is a quadratic function of $\boldsymbol{\omega}$ and $\boldsymbol{H}$. There are two preferential directions of $\boldsymbol{\omega}$ and $\boldsymbol{\eta}$ ($\boldsymbol{\eta}$ is the unit vector directed along OX_3). From two vectors $\boldsymbol{\eta}$, we can construct a scalar and a second-rank tensor. This also refers to the angular velocity $\boldsymbol{\omega}$. Therefore, the energy V_H is expressed through irreducible tensors as

$$V_H = \delta[B\omega^2 + (d/2)(\{\boldsymbol{\eta}_1 \otimes \boldsymbol{\eta}_1\}_2 \cdot \{\boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_1\}_2)],$$

where B and d are constants depending on the field harmonics and elastic constants of the body.

Using the relation $\{\boldsymbol{\eta}_1 \otimes \boldsymbol{\eta}_1\}_2 \cdot \{\boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_1\}_2 = (\boldsymbol{\eta} \cdot \boldsymbol{\omega})^2 - \omega^2/3$ [3], we can reduce the expression V_H to $V_H = \delta[\tilde{B}\omega^2 + (d/2)(\boldsymbol{\eta} \cdot \boldsymbol{\omega})^2]$. The generic form of the moment $\boldsymbol{M}(\boldsymbol{H})$ calculated using V_H is

$$\boldsymbol{M}(\boldsymbol{H}) = d(\boldsymbol{\eta} \cdot \boldsymbol{\omega})(\boldsymbol{\eta} \times \boldsymbol{\omega}). \tag{3.2}$$

Thus, the force function V in the approximation considered is found from the formula

$$V = \varepsilon a\omega^4 + \delta[\tilde{B}\omega^2 + (d/2)(\boldsymbol{\eta} \cdot \boldsymbol{\omega})^2]. \tag{3.3}$$

Owing to the presence of small parameters in (3.3), the kinetic energy of a perfectly rigid body is significantly greater than the maximum value of the force function. Since the moments of inertia are close, we assume that $I^0_{ii} = I^0 + \delta^* \tilde{I_{ii}}$ (I^0 is the moment of inertia of a sphere and δ^* is a small parameter of the same order as ε and δ). In the zero approximation, we obtain $\boldsymbol{K}^0 = \text{const}$, $\beta = \beta_0 = \text{const}$, $\gamma = \gamma_0 = \text{const}$, and $\dot{\alpha} = K^0/I^0$, i.e., $\omega = K^0/I^0$. Substituting $\boldsymbol{\omega}$ into (3.2) and (3.3), we obtain the following equations in the first approximation:

$$\begin{gathered}\frac{d\boldsymbol{K}^0}{dt} = \frac{\delta d}{(I^0)^2}\,(\boldsymbol{\eta} \times \boldsymbol{K}^0)(\boldsymbol{\eta} \cdot \boldsymbol{K}^0), \qquad \dot{\alpha} = K^0\Big(\frac{\cos^2\gamma}{I^0_{11}} + \frac{\sin^2\gamma}{I^0_{22}}\Big), \\ \dot{\beta} = K^0\Big(\frac{1}{I^0_{22}} - \frac{1}{I^0_{22}}\Big)\sin\beta\sin\gamma\cos\gamma, \qquad \dot{\gamma} = K^0\cos\beta\Big(\frac{1}{I^0_{33}} - \frac{\cos^2\gamma}{I^0_{11}} - \frac{\sin^2\gamma}{I^0_{22}}\Big).\end{gathered} \tag{3.4}$$

It follows from the first equation in (3.4) that $|\boldsymbol{K}^0| = \text{const}$ and $\boldsymbol{\eta} \cdot \boldsymbol{K}^0 = \text{const}$, i.e., the projection of the vector $\boldsymbol{K}^0$ onto the direction η remains constant.

Thus, in the approximation considered, the kinetic moment $\boldsymbol{K}^0$ whose magnitude remains constant precesses around the vector $\boldsymbol{\eta}$ with a constant angle of precession. The precession velocity is determined by the equality

$$\Omega_{\rm pr} = \frac{\delta d}{I_0^2}\,K^0\cos\rho = \frac{\delta d}{I_0^2}\,\boldsymbol{K}^0 \cdot \boldsymbol{\eta},$$

and the body performs free Euler–Poisson's motion around the kinetic moment vector. The precession velocity $\dot{\alpha}$ differs from the corresponding characteristics of free nutation of the body by a small quantity.

In conclusion, we make the following comments. Deformation of the rotor during its rotation may be considered (similarly to the London moment [6, 7] or disbalance [8]) as an inhomogeneity acting on the displacement gauge. The signal from the gauge changes the current in supporting coils, which leads to force actions on the rotor. Since there are correcting circuits in the control system, an accelerating or decelerating moment averaged over the revolution appears as a result of self-modulation of the suspension current.

The motion of an electrostatic gyroscope may be studied in a similar manner by substituting the vector $\boldsymbol{H}$ by the electric field strength $\boldsymbol{E}$ and changing the boundary conditions of the electrodynamic problems (1.1) and (1.2).

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